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Sets, Groups, and Mappings. An Introduction to Abstract Mathematics. By Andrew D. Hwang. American Mathematical Society, Pure and Applied Undergraduate Texts No. 39, 2019. Hardcover, pp. xv+304. Price USD 82.00. ISBN 9781470449322.

Massimo Nespolo

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Massimo Nespolo*

Université de Lorraine, CNRS, CRM2, Nancy, France. *Correspondence e-mail: massimo.nespolo@univ-lorraine.fr

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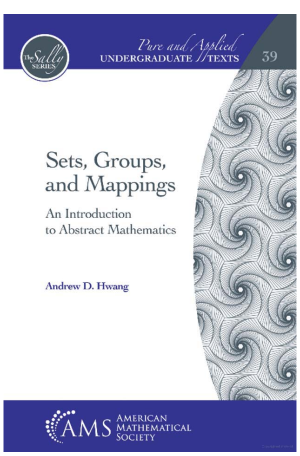

Abstract algebra, sometimes also called modern algebra, is the study of algebraic structures, *i.e.* sets on which act one or more operations. Depending on the nature and number of the operations, algebraic structures are divided into groups, rings, fields, vector space, modules and many others.

Crystallographers are familiar with groups, point groups and space groups in particular, which they often learn in a rather axiomatic application-driven approach. There are obvious limits to the 'it-works-in-this-way' learning strategy, as clearly shown by the difficulties some structural crystallographers face in reading off information from the *International Tables for Crystallography* (2016). An approachable textbook that gives the necessary background for a more in-depth understanding is clearly welcome. Crystallographers, and more generally scientists working with the structure of matter, come from very different backgrounds, and for some of them a textbook written by mathematicians for mathematics students, even at the undergraduate level, may be too hard to approach, not because the concepts are too complex, but because the way in which these concepts are presented is remote from what the potential reader is used to (which can be regarded as a language barrier).

This book by Andrew D. Hwang is a rare example of a textbook for mathematics students which, with minimal effort, is definitely suited for self-study by practically everybody. No special prerequisites are assumed for the reader, who is gently accompanied across the field with plenty of explanations and examples. The large number of worked-out examples (I have counted 285, of unequal length and difficulty) is certainly one of the qualities of this book. The algebra presented is pretty fundamental: hereafter I concentrate on why and how the crystallographer can take advantage of such a textbook.

After two brief introductions, *To the Instructor* and *To the Student*, the text develops in 18 chapters, plus an appendix and a detailed index. Each chapter ends with a series of exercises (467 overall). Additional material is available online at http://www.ams.org/publications/authors/books/postpub/amstext-39, where the author offers some interactive web applications related to the book, although not the solutions to the exercises.

The 18 chapters are not organized in a coarser division, although a study plan is proposed in the introduction *To the Instructor*. From our viewpoint, however, we can ideally gather the first five chapters into a preliminary subdivision, aimed at providing the reader with the basic language to approach the core of the subject.

Chapter 1 (13 pages) has the title *Logic and Proofs* and might appear a distant approach to the target. As a matter of fact, when studying group theory not as one's main subject, several concepts are often assumed to be known at least intuitively. Taking the time to rethink them avoids falling into some traps of reasoning. This chapter fits precisely that purpose. It starts by mentioning the Zermelo–Fraenkel set theory (also known as ZFC), the axiomatic system proposed in the early twentieth century to formulate a theory of sets free of paradoxes, which will run as an invisible thread to introduce the notion of a *set* in the following chapter (ZFC is not, however, discussed in detail in this book). The first chapter introduces and exemplifies the notions of abstract statement and negation, the logical connectives *and*, *or* and *implies*, with the emphasis on the *exclusive or* (*xor*: either one of two statements is true but not both), whose difference from the simple *or* (which includes the possibility of both statements being true) is often overlooked. This is followed by the notion of quantification (universal quantification: *for every*; existential: *it exists*), which are extensively used in the following.

Chapter 2 (17 pages) is *An Introduction to Sets* and does not present any difficulty for self-study. For most readers, this chapter will mainly be a refresh of well known concepts, with the addition of some details that are often missing in application-oriented group-theory textbooks. The definition of a set is pretty intuitive and immediately made concrete by a few examples. Other intuitive concepts, which are often skipped in group-theory textbooks used by crystallographers, are those of a *singleton* (a set containing a single element), a *universe* (a set containing all the sets one is going to deal with) and a *predicate* (the logical condition by which one selects elements of a universe to build its subsets). Sure, this can be considered a simple naming process, yet naming an object or an idea implies defining it and drawing the reader's attention to it and to the potential logical pitfalls. The most striking example is the non-existence of *the* universe (set of all sets), which would lead to Russell's paradox, to overcome which ZFC was introduced. A few pages are devoted to the analysis of the set of complex numbers, its geometric interpretation (Cartesian plane, Gaussian integers, *n*th roots of identity) and the operations on it. It is followed by a section on partitions, which leads to the notion of a power set (the set of all subsets).

Chapter 3 (13 pages) deals with *The Integers*. The set of integers is very often used to introduce, in a rather intuitive way, the notions of binary operations (operations that act on two elements of a set to produce another element of the same set) and the properties that possibly derive once the binary operation(s) are chosen: associativity, commutativity, distributivity, presence of the neutral element (the 'identity') and presence of the inverse of each element. Analysing a set whose properties are well known is an excellent exercise to formalize those properties in a more general and abstract way, so that they can later be applied – when they do apply – to sets which are less straightforward to grasp.

With Chapter 4 (24 pages) we get closer to the main subject, through the analysis of *Mappings and Relations*. Groups of interest to crystallographers are sets of isometries endowed with a binary operation which consists of the successive application of those isometries to a crystal structure. Mappings and relations are therefore concepts whose full understanding is indispensable to approach crystallographic groups. Basic notions like surjectivity, injectivity and bijectivity are explained with several examples. Particularly appreciated is the emphasis on details that may otherwise go unnoticed – or insufficiently noticed – by the student: the *uniqueness* of the result of a mapping, the imprecise wording 'one-to-one correspondence' for a bijection (it applies to an injection as well), and the variable results of the same mapping when the domains or codomains differ.

Chapter 5 (21 pages) introduces the methods of *Induction and Recursion*, to be used in the demonstration in the following chapters. Many worked-out examples allow the reader to become familiar with the methods.

Once these introductory chapters are sufficiently digested, the reader can approach with confidence the core of the subject, which starts with Chapter 6 (13 pages) on *Binary Operations*. As expected in a textbook for mathematicians, the emphasis is put on a mapping being a binary operation, or not, depending on the set on which it acts: although elementary, this notion is usually overlooked in group-theory textbooks addressing structural scientists. Experience shows that spending a bit of time on the definition of binary operations avoids potential misunderstandings in the study of groups.

Groups are the subject of Chapter 7 (18 pages), which also deals with subgroups. From the very beginning the difference between a set and a group (a set endowed with an associative binary operation having a neutral element and including the inverse of the element) is emphasized. It may sound obvious, but so many texts forget or overlook this fundamental difference and fall into linguistic traps that result in confusion for the reader. Nevertheless, some inconsistencies do occur in a couple of examples (Sections 7.27 and 7.50), where *sets* are presented as *subgroups*. As usual when dealing with texts from another field, one may find terms used with a different definition. An example is the 'honeycomb lattice' (p. 118), which corresponds to the *hp* (two-dimensional hexagonal) lattice in crystallography.

The next three chapters, *Divisibility and Congruences* (10 pages), *Primes* (10 pages) and *Multiplicative Inverses of Residues Classes* (13 pages), deal with the division of the integers as a convenient and easily approachable way of introducing fundamental concepts like divisibility (distinct from invertibility!), congruence and equivalence, and residue classes (prerequisite to understanding cosets in groups), through the analysis of number of examples of possible traps into which the reader might fall (*e.g.* the cancellation law does not hold mod *n*). Particularly inspiring are the examples in Chapter 10.

Chapter 11 (21 pages) introduces *Linear Transformations*, with particular reference to the Cartesian plane. The full categorization of linear transformations is not given, and the emphasis is placed on scaling and isometries, which is perfectly fine considering the target is groups. A terminological objection can be moved to the statement (p. 163) that a 'half-turn about the origin' would be equivalent to a 'reflection about the origin': the result of the action is the same, but the nature of the operation (first kind versus second kind) is not.

Chapter 12 (11 pages) introduces *Isomorphism*, an extremely important concept which in crystallography we usually present after that of homomorphism (here postponed to Chapter 16). It is followed by a chapter on *The Symmetric Group* (15 pages), the group of permutations which is not always explicitly presented in group-theory books for crystallographers. This is a surprising choice, considering that every group is isomorphic to a subgroup of the symmetric group! Not every reader may need to follow all the demonstrations, but the examples are all insightful. Chapter 14 (15 pages) presents *Examples of Finite Groups*, in particular symmetry groups of polyhedra, whose importance for crystallographers cannot be underestimated.

With Chapter 15 (11 pages) we arrive at the fundamental concept of *Cosets* and normal subgroups. The examples are abstract groups but are quite straightforward to follow. This is followed by a chapter on *Homomorphisms* (15 pages), which play a role of paramount importance in representation theory (outside the scope of this book). *Quotient groups*, *centre* and *centralizer* are dealt with briefly, before moving to the three isomorphism theorems.

Chapter 17 (22 pages) deals with *Group Actions* and deserves to be studied carefully. In fact, experience shows that the difference between a group itself and the result of its action is not always fully realized by the final user. The concepts of *orbit* and *stabilizer* are introduced here, which have direct application to the symmetry of crystal structures as *crystallographic orbits* (sets of equivalent atomic positions) and *site-symmetry groups*, respectively. Sylow theorems are then introduced in order to present a brief classification of finite groups.

The last chapter (22 pages) deals with *Euclidean Geometry*, where we find the basic elements of crystallographic groups in two dimensions. The term 'affine isometries' (p. 283) sounds a bit surprising to me, an isometry being a special case (distance-preserving) of affine motion.

The book ends with a short appendix (2 pages) about *Euler's Formula* and an index which is detailed (it spans six pages) yet misses a few entries, like cosets (to which a whole chapter is devoted) and affine motion (p. 283).

Overall, this book is written in a pedagogical student-oriented way and without a doubt represents an interesting teaching resource for those who need to lecture on crystallographic group theory. We are not short of textbooks, but few are worth mentioning among the plethora published recently that are more or less accurate clones of older classical textbooks. At the same time, even excellent textbooks often miss a more fundamental and general introductory part, perhaps considered too abstract, whereas it is absolutely necessary to push the understanding beyond the level of an instruction manual. When trying to cope for this insufficiency through browsing a textbook of abstract algebra, the instructor is however often discouraged by the effort necessary to extract the necessary concepts and translate them into our language of crystallographers. Hwang's book sits in between: it has the great advantage of being approachable without too much effort by practically anybody, although it does not cover the full subject (it is not meant to). Actually, in the analysis of cases and examples, more general algebraic structures than groups are briefly presented without being mentioned: loops, semigroups, monoids *etc.* The decision to stick to groups, while simply showing that the features of a group may or may not be realized depending on the choice of the set and of the binary operation, is probably appropriate to avoid distracting the reader from the main target (remember, the textbook addresses undergraduates). Nevertheless, an additional short chapter recalling the exceptions mentioned in the text and showing that they define different algebraic structures would perhaps have been useful as a sort of *appetizer* for those who wish to go beyond.

If one wants to find a defect, perhaps one could mention that some concepts which could have been developed in the text are confined to exercises, whose solutions are not available. For example, the quaternion group is introduced in Exercise 11.33 and recalled in Example 14.21, but is absent from the text.

Overall, I recommend this book, both to lecturers teaching introductory group theory to structural scientists, and to students who will undoubtedly find it inspiring to fill the holes left in their constantly shrinking basic education programme.

References

International Tables for Crystallography (2016). Vol. A, *Space-group symmetry*, 6th ed., edited by M. I. Aroyo. Chichester: Wiley.